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Some Integrals Involving Multivariable H – Functions & M- Series

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In this paper, the author presented certain integral formulas (finite and infinite) plays an important role in the study of generalized hypergeometric function. There are a number of paper on such results. We establish certain definite integrals of involving the product of multivariable H-function & M-series. The results derived here and basic in natural and many include a number of known and new results as particular cases.

Keywords: Generalized hypergeometric function, Finite integral, M- Series, Multivariable H-functions.

The multivariable H-function which was introduced and investigated by Srivastava & Panda[5] in term of a multiple Mellin -Bernes type contour integral as

$$H[z_1,\ldots,z_r] = H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle|\begin{matrix}(a_j;\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,\ldots,\beta_j^r)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\left\{\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i}\right\}d\xi_1\ldots d\xi_r, \qquad \ldots(1)$$

Where $\omega=\sqrt{-1}$; and

$$\psi(\xi_1,\ldots,\xi_r)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\Sigma_{i=1}^{r}\alpha_j^{(i)}\xi_i)}{\prod_{j=n+1}^{p_i}\Gamma(a_j-\Sigma_{i=1}^{r}\alpha_j^{(i)}\xi_i)\Pi_{j=1}^{q_i}\Gamma(1-b_j+\Sigma_{i=1}^{r}\beta_j^{(i)}\xi_i)} \qquad \ldots(2)$$

$$\phi_i(\xi_i)=\frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i)\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}\xi_i)}{\Pi_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}\xi_i)\,\Pi_{j=m_i+1}^{q}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_i)} \quad (i=1,\ldots,r); \qquad \ldots(3)$$

And $L_j=L_{\omega\tau_j\infty}$ represents the contours which start at the point $\tau_j-\omega\infty$ and terminate at the points $\tau_j+\omega\infty$ with $\tau_j\in\Re=(-\infty,\infty)(j=1,\ldots,r)$.

Result required in the sequel

The well known M-series, which is a particular case of $\overline{H}$ -function introduced by Inayat Hussain[4] and is defined by means of the following series expansion:

$$_PM_Q^{\tau}(\mathrm{A}_1,\ldots,\mathrm{A}_p;\mathrm{B}_1,\ldots,\mathrm{B}_Q;\mathrm{X})=\sum_{k=0}^{\infty}\frac{(\mathrm{A}_1)_k\ldots(\mathrm{A}_p)_k}{(\mathrm{B}_1)_k\ldots(\mathrm{B}_Q)_k}\frac{\mathrm{X}^k}{\Gamma(\tau k+1)} \qquad \ldots(4)$$

Provided that $\tau\in C$, $R(\tau)>0$, $(A_j)_k$ $(B_j)_k$ are pochammer symbols.

(A) when $\tau=1$, we have,

$$_PM_Q^1(A_1,\ldots,A_p;B_1,\ldots,B_Q;\mathrm{X})=\sum_{k=0}^{\infty}\frac{(A_1)_k\ldots(A_p)_k}{(B_1)_k\ldots(B_Q)_k}\frac{\mathrm{X}^k}{\Gamma(k+1)}={}_PF_Q(X) \qquad \ldots(5)$$

Thus the series $_\mathrm{P}\mathrm{M}_\mathrm{Q}^1(\mathrm{X})$ reduced to the generalized hypergeometric function

(B) $_0M_0^{\tau}$ i.e. no upper or lower parameters .

$$_0M_0^{\tau}(\ldots;\ldots;\mathrm{X})=\sum_{k=0}^{\infty}\frac{(X)^k}{\Gamma(\tau k+1)} \qquad \ldots(6)$$

Thus the series reduced to the Mittag-Leffler function.

Proper Integral

The integral to be established in this paper are:

$$\int_0^1\frac{x^{\rho-1}(1-x)^{\sigma-1}}{(x+r)^{\rho+\sigma}}{}_PM_Q^{\tau}\left[(A_P):(B_Q):y\frac{x^u(1-x)^v}{(x+r)^{u+v}}\right]$$
$$\times\mathrm{H}^{0,\mathrm{n}:\mathrm{m}_1,\mathrm{n}_1;\ldots;\mathrm{m}_r,\mathrm{n}_r}_{\mathrm{p},\mathrm{q}:\mathrm{p}_1,\mathrm{q}_1;\ldots;\mathrm{p}_r,\mathrm{q}_r}\left[\begin{matrix} z_1x^{\eta_1} \\ \vdots \\ z_rx^{\eta_r}\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j^1,\ldots,\alpha_j^{(r)})_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,\ldots,\beta_j^{(r)})_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]dx$$

$$=\frac{1}{r^{\rho}(1+r)^{\rho+\sigma}}\sum_{k=0}^{\infty}M(k)$$

$$\times H^{0,n+2:m_1,n_1;\ldots;m_r,n_r}_{p+2,q+1:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}\frac{z_1}{r^{\mu_1}(1+r)^{\mu_1+\upsilon_1}} \\ \vdots \\ \frac{z_r}{r^{\mu_r}(1+r)^{\mu_r+\upsilon_r}}\end{matrix}\middle|\begin{matrix}(1-\rho-uk;\prod_{i=1}^{r}\mu_i),(1-\sigma-vk;\prod_{i=1}^{r}\upsilon_i),(a_j;\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,\ldots,\beta_j^{(r)})_{1,q},(1-\rho-\sigma-uk-vk;\prod_{i=1}^{r}\mu_i+\upsilon_i):(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]$$

$$\times r^{-uk}(1+r)^{-(u+v)k} \qquad \ldots(7)$$

*Corresponding author (E-mail: priyanka103gupta@gmail.com) where,

$$\mathrm{M(k)} = \frac{\Pi_{j=1}^{P}(A_j)_k}{\Pi_{j=1}^{Q}(B_j)_k}\frac{y^k}{\Gamma(\tau k+1)} \qquad ...(8)$$

and

$$X^{\eta_i} = \frac{x^{\mu_i}(1-x)^{\upsilon_i}}{(x+r)^{\mu_i+\upsilon_i}} \qquad ...(9)$$

(i)

$\mu \ge 0, \upsilon \ge 0$ (not both zero simultaneously) $\Omega > 0, \delta \ge 0, |\text{large}| < \frac{\pi}{2}\Omega$ $|\,\Omega$ *and* δ *are defined in* {2}|

(ii) ς, η *and rare positive* constant.

(iii)

$p \le q$, *or* $p = q+1$ *and* $|y| < 1$, [*none of* B_1 $(j = 1, ..., q)$ *is a negative* integer or zero.

(IV)

$$\mathrm{Re}(\rho) + \mu \min_{1\le j\le m}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right] > 0 \; and \; \mathrm{Re}(\sigma) + \upsilon \min_{1\le j\le m}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right] > 0,$$

Proof

To establish (7), first we expressing the Multi-variable H-function involved in its left and side in terms of Mellin-Barnes type of contour integral using equation (1) and the M-series in series form given by equation (4). Interchanging the order of integration and Summation (which is permissible under the condition stated with (7)), we get

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}...\int_{L_r}\psi(\xi_1,...,\xi_r)\left\{\prod_{i=1}^{r}\phi_i(\xi_i)\right\}\sum_{k=0}^{\infty}M(k)$$
$$\left\{\int_0^1 \frac{x^{\rho+uk+\mu_i\xi_i-1}(1-x)^{\sigma+vk+\upsilon_i\xi_i-1}}{(x+r)^{\rho+\sigma+uk+vk+\mu_i\xi_i+\upsilon_i\xi_i}}dx\right\} z_i^{\xi_i}\, d\xi_1,...,d\xi_r,$$

{M(k) is given by eqn. (8)}. Evaluating the inner integral with the help of the following well-known formula from Spiegel[4]:

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+r)^{m+n}}dx = \frac{B(m,n)}{r^m(1+r)^{m+n}} \qquad ...(10)$$

Where m, n and rare positive constants. After simplifying and interpreting in view of (1) we arrive at the desired result.

Special cases

(A) In eqn. (7), when $\tau = 1$, we have,

$$\int_0^1 \frac{x^{\rho-1}(1-x)^{\sigma-1}}{(x+r)^{\rho+\sigma}}\,{}_PF_Q[y\frac{x^u(1-x)^v}{(x+r)^{u+v}}]$$
$$\times \mathrm{H}^{0,n:m_1,n_1;...;m_r,n_r}_{p,q:p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1x^{\eta_1} \\ \vdots \\ z_rx^{\eta_r}\end{matrix}\middle| \begin{matrix}(a_j;\alpha_j^1,...,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};...;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,...,\beta_j^r)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};...;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]dx$$
$$\times \mathrm{H}^{0,n+2:m_1,n_1;...;m_r,n_r}_{p+2,q+1:p_1,q_1;...;p_r,q_r}\left[\begin{matrix} \frac{z_1}{r^{\mu_1}(1+r)^{\mu_1+\upsilon_1}} \\ \vdots \\ \frac{z_r}{r^{\mu_r}(1+r)^{\mu_r+\upsilon_r}}\end{matrix}\middle| \begin{matrix}(1-\rho-uk;\prod_{i=1}^{r}\mu_i),(1-\sigma-vk;\prod_{i=1}^{r}\upsilon_i),(a_j;\alpha_j^1,...,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};...;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,...,\beta_j^r)_{1,q},(1-\rho-\sigma-uk-vk;\prod_{i=1}^{r}\mu_i+\upsilon_i):(d_j^1,\delta_j^1)_{1,q_1};...;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]$$
$$\times r^{-uk}(1+r)^{-(u+v)k} \qquad ...(11)$$

where,

$$F(k) = \frac{\Pi_{j=1}^{P}(A_j)_k}{\Pi_{j=1}^{Q}(B_j)_k}\frac{y^k}{\Gamma(k+1)} \qquad ...(12)$$

Thus the series $M(k)$ reduces to generalized hypergeometric functions.

(B *) In eqn. (7), when* ${}_0\mathrm{M}_0^{\tau}$ *i.e. no upper or lower parameters we have,*

$$\int_0^1 \frac{x^{\rho-1}(1-x)^{\sigma-1}}{(x+r)^{\rho+\sigma}}\,{}_0\mathrm{M}_0^{\tau}(...:...;y\frac{x^u(1-x)^v}{(x+r)^{u+v}})$$
$$\times \mathrm{H}^{0,n:m_1,n_1;...;m_r,n_r}_{p,q:p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1x^{\eta_1} \\ \vdots \\ z_rx^{\eta_r}\end{matrix}\middle| \begin{matrix}(a_j;\alpha_j^1,...,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};...;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,...,\beta_j^r)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};...;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]dx$$
$$= \frac{1}{r^{\rho}(1+r)^{\rho+\sigma}}\sum_{k=0}^{\infty}\frac{(y)^k}{\Gamma(\tau k+1)}$$
$$\times \mathrm{H}^{0,n+2:m_1,n_1;...;m_r,n_r}_{p+2,q+1:p_1,q_1;...;p_r,q_r}\left[\begin{matrix} \frac{z_1}{r^{\mu_1}(1+r)^{\mu_1+\upsilon_1}} \\ \vdots \\ \frac{z_r}{r^{\mu_r}(1+r)^{\mu_r+\upsilon_r}}\end{matrix}\middle| \begin{matrix}(1-\rho-uk;\prod_{i=1}^{r}\mu_i)(1-\sigma-vk;\prod_{i=1}^{r}\upsilon_i),(a_j;\alpha_j^1,...,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};...;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,...,\beta_j^r)_{1,q},(1-\rho-\sigma-uk-vk;\prod_{i=1}^{r}\mu_i+\upsilon_i):(d_j^1,\delta_j^1)_{1,q_1};...;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]$$
$$\times r^{-uk}(1+r)^{-(u+v)k} \qquad ...(13)$$

Thus the series M(k) reduced to the Mittag-Leffler function.

Conclusion

The Multivariable H-function and M – Series described is basic in nature and henceforth, we could get various other special functions such as Whittaker function, Meijer's G-function, Wright's generalized Bessel function, Fox's H-function, Wright's generalized hypergeometric function, Mac-Robert's E-function, Bessel function of first kind, generalized hypergeometric function, modified Bessel function, exponential function, binomial function etc. on specializing the parameters of this function, and therefore, various unified integral presentations are possible to obtained as special cases.

References:

1 Mishra P P & Shrivastava R, *Reflections des ERA-IMS*, 6 (2011) 281.

2 Sharma C K & Shrivastava S, *Proc Nm Acad Sci India*, 62 (1992) 235.

3 Prudnikov A P, Brychkov Y & Marichev O I, integrals and Series, (More Special Functions, Gordon and Breach, New York), 3, NJ (1990).
4 Hussian A A & Inayat, *J Phys A: Math Gen,* 20 (1987) 4119.
5 Srivastava H M, Gupta K C & Goyal S P, The H-Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi, 1982.
6 Spiegel & Murray P, Theory and problems of complex variables, Schaum's outline Series, (McGraw Hill book Company, Singapore) 1981.
7 Mathai A M & Saxena R K, The H- function with Applications in statistics and other Disciplines, (Wiley Easterm Limited, New Delhi,) 1978.
8 Srivastava H M & Panda R, *J Reine Angew Math,* 283/284 (1976) 265.
9 Srivastava H M & Panda R, *Math Univ St Paul,* 25 (1976) 167.
10 Slater L J, *Generalized Hypergeometric functions,* (Cambridge University Press), 1966.
11 Mittag –Leffler G M, *C R Acad, Sci Paris (*Ser.II) 137 (1903) 554.