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Relativistic solution of Eckart plus Hulthen potentials in the presence of spin and pseudospin symmetry

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In this paper Dirac equation has been solved approximately for Eckart plus Hulthen potentials with spin symmetry and pseudospin symmetry for $k\neq0$. The formula method has been used to obtain the energy Eigen-values and wave functions. The energy Eigen-values and wave functions have also been discussed for the special case of modified Eckart plus Hulthen potentials for the spin and pseudospin symmetry with formula method. To show the accuracy of the present model, some numerical values of the energy levels have been shown.

Keywords: Dirac equation, Eckart potential, Hulthen potential, Formula method, Spin and pseudospin symmetry

1 Introduction

In quantum mechanics, it is well known that the analytical solutions play a fundamental role, because, this solution usually contain all the necessary information about the quantum mechanical model under investigation[1,2]. Therefore, one of the interesting problems in nuclear and high energy physics is to obtain an analytical solution of the Klein - Gordon, Duffin- Kemmer - Petiau and Dirac equations for mixed vector and scalar potentials[3]. The study of relativistic effects has been always useful in some quantum mechanical systems[4,5]. Therefore, the Dirac equation has become the most appealing relativistic wave equations for spin-1/2 particles. For example, in the relativistic treatment of nuclear phenomena the Dirac equation is used to describe the behaviour of the nuclei in the nucleus and also in solving many problems of high-energy physics and chemistry. For this reason, it has been used extensively to study the relativistic heavy ion collisions, heavy ion spectroscopy and more recently in laser–matter interaction (for a review, see[6] and references therein) and condensed matter physics[7,8].

The idea about spin symmetry and pseudo-spin symmetry with the nuclear shell model has been introduced in 1969 by Arima *et al.*[1], Hecht and Adler[2]. Spin and pseudospin symmetries are SU(2) symmetries of a Dirac Hamiltonian with vector and scalar potentials. They are realized when the difference, $\Delta(r)=V(r)-S(r)$, or the sum, $\Sigma(r)=V(r)+S(r)$, are constants. The near realization of these symmetries may explain degeneracy in some heavy meson spectra (spin symmetry) or in single-particle energy levels in the nuclei (pseudospin symmetry), when these physical systems are described by relativistic mean-field theories (RMF) with scalar and vector potentials[9-11]. The kind of various methods has been used for the analytical solutions of the Klein–Gordon equation and Dirac equation such as the super symmetric quantum mechanics[12-14], asymptotic iteration method[15,16] (AIM), factorization method[17,18], Laplace transform approach[19], GPS Method[20,21] and the path integral method[22-24], Nikiforov-Uvarov method[25-27] and others.

The Klein-Gordon and Dirac wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics with some typical potential by using different methods[28]. For example, Kratzer potential[29,30], Woods-Saxon potential[31,32], Scarf potential[33,34], Hartmann potential[35,36], Rosen Morse potential[37,38], Hulthen potential[39] and Eckart potential[40,41].

In this paper, we attempt to solve analytically Dirac wave equation for $k\neq0$ with Eckart plus Hulthen potentials for the spin and pseudospin symmetry by using the Formula method. We also discuss the special case of modified Eckart plus Hulthen potentials for the spin and pseudospin symmetry.

2 Review of Formula Method

The formula method has been used to solve the Schrodinger, Dirac and Klein-Gordon wave equations

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for a certain kind of potential. In this method the differential equations can be written as follows[42,43]:

$$\psi_n''(s) + \frac{(k_1-k_2 s)}{s(1-k_3 s)}\psi_n'(s) + \frac{(\eta_2 s^2+\eta_1 s+\eta_0)}{s^2(1-k_3 s)^2}\psi_n(s) = 0 \quad \dots (1)$$

For a given Schrödinger-like equation in the presence of any potential model which can be written in the form of Eq. (1), the energy Eigen-values and the corresponding wave function can be obtained by using the following formulas, respectively[44]:

$$k_4 + k_5 = \left[\frac{1-2n}{2} - \frac{1}{2k_3}\left(k_2 - \sqrt{(k_3-k_2)^2 - 4\eta_2}\right)\right] \quad \dots (2)$$

or more explicitly as:

$$\left[\frac{k_4^2 - k_5^2 - \left[\frac{1-2n}{2} - \frac{1}{2k_3}\left(k_2 - \sqrt{(k_3-k_2)^2-4\eta_2}\right)\right]^2}{2\left[\frac{1-2n}{2} - \frac{1}{2k_3}\left(k_2 - \sqrt{(k_3-k_2)^2-4\eta_2}\right)\right]^2}\right]^2 \dots$$
$$-k_5^2 = 0 \,.\, k_3 \neq 0 \quad \dots (3)$$

$$\psi_n(s) = N_n s^{k_4}(1-k_3 s)^{k_5}\, {}_2F_1\left(-n.n + 2(k_4 + k_5) + \frac{k_2}{k_3} - 1; 2k_4 + k_1 . k_3 s\right) \quad \dots (4)$$

where,

$$\begin{cases} k_4 = \frac{(1-k_1)+\sqrt{(1-k_1)^2-4\eta_0}}{2} \\ k_5 = \frac{1}{2} + \frac{k_1}{2} - \frac{k_2}{2k_3} + \sqrt{\left[\frac{1}{2} + \frac{k_1}{2} - \frac{k_2}{2k_3}\right]^2 - \left[\frac{\eta_2}{k_3^2} + \frac{\eta_1}{k_3} + \eta_0\right]} \end{cases} \quad \dots (5)$$

And N_n is the normalization constant. In special case where $k_3 \to 0$ the energy Eigen-values and the corresponding wave function can be obtained as[44]:

$$\left[\frac{\eta_1 - k_4 k_2 - n k_2}{2k_4 + k_1 + 2n}\right]^2 - k_5^2 = 0 \quad \dots (6)$$

$$\psi_n(s) = N_n s^{k_4} \exp(-k_5 s)\, {}_1F_1(-n; 2k_4 + k_1; (2k_5 + k_2)s) \quad \dots (7)$$

where,

$$\begin{cases} k_4 = \frac{(1-k_1)+\sqrt{(1-k_1)^2-4\eta_0}}{2} \\ k_5 = -\frac{k_2}{2} + \sqrt{\left(\frac{k_2}{2}\right)^2 - \eta_2} \end{cases} \quad \dots (8)$$

The solution provides a valuable means for checking and improving models and numerical methods introduced for solving complicated quantum systems.

3 Basic Dirac Equations

In the relativistic description, the Dirac equation of a single-nucleon with the mass moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ can be written as[45]:

$$\left[-i\hbar c\hat{\alpha}\cdot\hat{\nabla} + \hat{\beta}\left(Mc^2 + S(r)\right)\right]\psi_{n_r.k} = [E - V(r)]\psi_{n_r.k} \quad \dots (9)$$

Where E is the relativistic energy, M is the mass of a single particle and α and β are the 4×4 Dirac matrices. For a particle in a central field, the total angular momentum J and $\hat{K} = -\hat{\beta}\left(\hat{\alpha}\cdot\hat{L} + \hbar\right)$ commute with the Dirac Hamiltonian where L is the orbital angular momentum. For a given total angular momentum j, the Eigen values of the $\hat{K}$ are $k=\pm(j+1/2)$ where negative sign is for aligned spin and positive sign is for unaligned spin. The wave-functions can be classified according to their angular momentum j and spin-orbit quantum number k as follows:

$$\psi_{n_r.k}(r.\theta.\varphi) = \frac{1}{r}\begin{bmatrix} F_{n_r.k}(r)Y_{jm}^{l}(\theta.\varphi) \\ iG_{n_r.k}(r)\, Y_{jm}^{\hat{l}}(\theta.\varphi) \end{bmatrix} \quad \dots (10)$$

Where $F_{n_r.k}(r)$ and $G_{n_r.k}(r)$ are upper and lower components, $Y_{jm}^{l}(\theta.\varphi)$ and $Y_{jm}^{\hat{l}}(\theta.\varphi)$ are the spherical harmonic functions. n_r is the radial quantum number and m is the projection of the angular momentum on the z axis. The orbital angular momentum quantum numbers l and $\hat{l}$ represent to the spin and pseudospin quantum numbers. Substituting Eq. (10) into Eq. (9), we obtain couple equations for the radial part of the Dirac equation as follows:

$$\begin{cases} \left(\frac{d}{dr} + \frac{K}{r}\right)F_{n_r.k}(r) = \frac{1}{\hbar c}[Mc^2 + E - \Delta(r)]G_{n_r.k}(r) \\ \left(\frac{d}{dr} - \frac{K}{r}\right)G_{n_r.k}(r) = \frac{1}{\hbar c}[Mc^2 - E + \Sigma(r)]F_{n_r.k}(r) \end{cases} \quad \dots (11)$$

Where $\Delta(r)=V(r)-S(r)$ and $\sum(r)=V(r)+S(r)$ are the difference and the sum of the potentials $V(r)$ and $S(r)$, respectively.

Under the condition of the spin symmetry, i. e., $\Delta(r)=0$, Eq. (11) reduces into:

$$\left(-\frac{d^2}{dr^2} + \frac{k(k+1)}{r^2} + \frac{1}{\hbar^2c^2}[Mc^2 + E][Mc^2 - E + \Sigma(r)]\right)F_{n_r.k}(r) = 0 \quad \dots (12)$$

Under the condition of the pseudospin symmetry, i.e., $\sum(r)=0$ Eq. (12) turns to the following form:

$$\left(-\frac{d^2}{dr^2}+\frac{k(k-1)}{r^2}+\frac{1}{\hbar^2c^2}[Mc^2-E][Mc^2+E-\Delta(r)]\right)G_{n_r,k}(r)=0 \quad \dots (13)$$

We consider bound state solutions that demand the radial components[45] satisfying $F_{n_r,k}(0)=G_{n_r,k}(0)=0$, and $F_{n_r,k}(\infty)=G_{n_r,k}(\infty)=0$.

4 Dirac Equations

4.1 Solution of Dirac Equations for spin symmetric

Under the condition of the spin symmetry, i.e., $\Delta(r)=0$, the upper component Dirac equation can be written as:

$$\left(-\frac{d^2}{dr^2}+\frac{k(k+1)}{r^2}+\frac{1}{\hbar^2c^2}[Mc^2+E][Mc^2-E+\Sigma(r)]\right)F_{n_r,k}(r)=0 \quad \dots (14)$$

The Eckart[40,41] plus Hulthen potentials[39] is defined as:

$$V(r)=q_1\mathrm{cosech}^2(\alpha r)-q_2\coth(\alpha r)+\frac{v_0}{(1-e^{-2\alpha r})}-\frac{v_1}{(1-e^{-2\alpha r})^2} \quad \dots (15)$$

and the Eq. (15) can be rewritten in the exponential form as:

$$V(r)=4q_1\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})}+\frac{v_0}{(1-e^{-2\alpha r})}-\frac{v_1}{(1-e^{-2\alpha r})^2} \quad \dots (16)$$

Where the parameters q_1, q_2, v_0 and v_1 are positive and real parameters, these parameters describe the depth of the potential well, and the parameter α is related to the range of the potential.

In Fig. 1(a,b), we show the behaviour of the Eckart plus Hulthen potentials as a function of r(fm) for three screening parameter values α= 0.12, 0.25, 0.45 fm^{-1} by taking the strength parameters $q_1=q_2=v_0=v_1=0.8$ and $q_1=v_0=0$, $q_2=v_1=0.8$. It is seen that the potential strength decreases with the increasing of the screening parameter value.

Under the condition of the spin symmetry sum of the potentials $V(r)$ and $S(r)$ can be written as:

$$\Sigma(r)=8q_1\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-2q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})}+\frac{2v_0}{(1-e^{-2\alpha r})}-\frac{2v_1}{(1-e^{-2\alpha r})^2} \quad \dots (17)$$

By substituting Eq. (17) into Eq. (14), we obtain the upper radial equation of Dirac equation as:

$$\left\{\frac{d^2}{dr^2}+\frac{(E^2-M^2c^4)}{\hbar^2c^2}-\frac{(E+Mc^2)}{\hbar^2c^2}\left[8q_1\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-2q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})}+\frac{2v_0}{(1-e^{-2\alpha r})}-\frac{2v_1}{(1-e^{-2\alpha r})^2}\right]-\frac{k(k+1)}{r^2}\right\}F_{n_r,k}(r)=0 \quad \dots (18)$$

Equation (18) is exactly solvable only for the case of $k=0,-1$. In order to obtain the analytical solutions of Eq. (18), we employ the improved pekeris approximation and replace the spin–orbit coupling term with the expression[46] that is valid for $\alpha r<<1$:

$$\frac{k(k+1)}{r^2}\approx\frac{k(k+1)4\alpha^2e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad \dots (19)$$

Using the transformations $=\exp(-2\alpha r)$ Eq. (18) brings into the form:

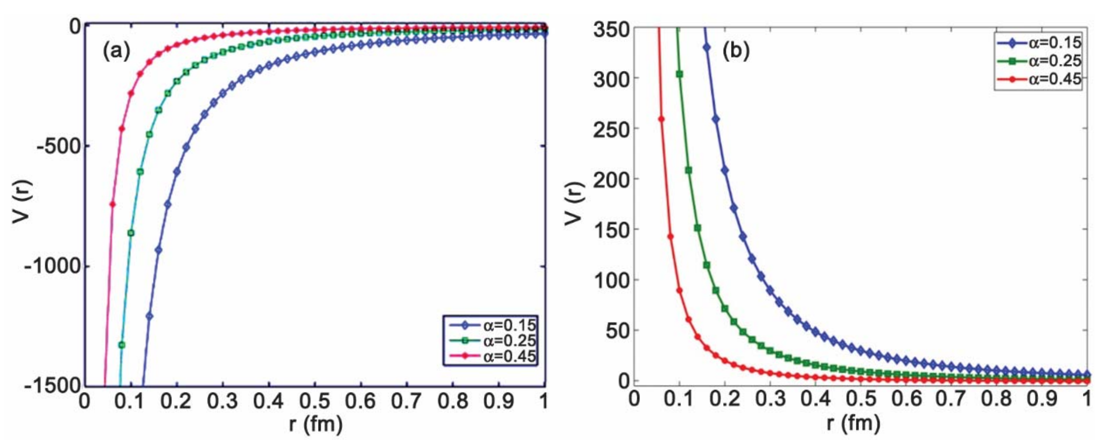

Fig. 1 — The behaviour of the Eckart plus Hulthen potentials with r(fm) for different values of the screening parameter α and taking (a) $q_1=v_0=0$ and $q_2=v_1=0.8$ and (b) $q_1=q_2=v_0=v_1=0.8$.

$$F''_{n_r.k}(s) + \frac{(1-s)}{s(1-s)} F'_{n_r.k}(s) + \frac{1}{s^2(1-s)^2}[\eta_2 s^2 + \eta_1 s + \eta_0] F_{n_r.k}(s) = 0 \quad \ldots (20)$$

Where the parameters η_2, η_1 and η_0 are considered as follows:

$$\begin{cases} \eta_2 = \frac{\delta}{4\alpha^2} - \frac{\gamma}{2\alpha^2} q_2 \\ \eta_1 = -\frac{\delta}{2\alpha^2} + \frac{\gamma}{2\alpha^2}(v_0 - 4q_1) - k(k+1) \\ \eta_0 = \frac{\delta}{4\alpha^2} + \frac{\gamma}{2\alpha^2}(q_2 - v_0 + v_1) \end{cases} \quad \ldots (21)$$

where, $\delta = \frac{(E^2 - M^2c^4)}{\hbar^2 c^2}$ and $\gamma = \frac{(E + Mc^2)}{\hbar^2 c^2}$

Now by comparing Eq. (20) with Eq. (1), we can easily obtain the coefficients k_i (i = 1, 2, 3) as follows:

$$k_1 = k_2 = k_3 = 1 \quad \ldots (22)$$

The values of the coefficients k_i (i = 4, 5) are also found from Eq. (5) as below:

$$\begin{cases} k_4 = \sqrt{-\eta_0} \\ k_5 = \frac{1}{2} + \sqrt{\frac{1}{4} - [\eta_2 + \eta_1 + \eta_0]} \end{cases} \quad \ldots (23)$$

Thus, by the use of energy equation (Eq. (2)) for energy Eigen-values, we find:

$$(2n+1) + \sqrt{\frac{2\gamma}{\alpha^2}(v_0 - q_2 - v_1) - \frac{\delta}{\alpha^2}} + \sqrt{(2k+1)^2 - \frac{2\gamma}{\alpha^2}(v_1 - 4q_1)} - \sqrt{\frac{2\gamma}{\alpha^2} q_2 - \frac{\delta}{\alpha^2}} = 0 \quad \ldots (24)$$

And using Eq. (21) we can obtain the energy Eigen-values equation, in closed form, as:

$$(2n+1) + \frac{\sqrt{(E+Mc^2)}}{\hbar c\alpha}\left[\sqrt{2(v_0 - v_1 - q_2) - (E - Mc^2)} - \sqrt{2q_2 - (E - Mc^2)}\right] + \sqrt{(2k+1)^2 - \frac{2(E+Mc^2)}{\hbar^2 c^2 \alpha^2}(v_1 - 4q_1)} = 0 \quad \ldots (25)$$

Let us find the corresponding wave functions. In reference to Eq. (4) and Eq. (23), we can obtain the upper wave function as:

$$F_{n_r.k}(r) = N(e^{-2\alpha r})^{(\sqrt{-\eta_0})}(1 - e^{-2\alpha r})^{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \eta_2 + \eta_1 + \eta_0}\right)} {}_2F_1\left(-n.n + 2\left(\sqrt{-\eta_0} + \frac{1}{2} + \sqrt{\frac{1}{4} + \eta_2 + \eta_1 + \eta_0}\right); 2\sqrt{-\eta_0} + 1.e^{-2\alpha r}\right) \quad \ldots (26)$$

where N is the normalization constant, on the other hand, the lower component of the Dirac spinor can be calculated from Eq. (27) as:

$$G_{n_r.k}(r) = \frac{\hbar^2 c^2}{(E + Mc^2)}\left(\frac{d}{dr} + \frac{k}{r}\right) F_{n_r.k}(r) \quad \ldots (27)$$

We have obtained the energy Eigen-values and the wave function of the radial Dirac equation for Eckart plus Hulthen potentials with the spin symmetry for $k \neq 0$.

4.2 Solution of Dirac equations for pseudospin symmetric

For the pseudospin symmetry, i.e., $\sum(r) = 0$ the lower component Dirac equation can be written as:

$$\left(-\frac{d^2}{dr^2} + \frac{k(k-1)}{r^2} + \frac{1}{\hbar^2 c^2}[Mc^2 - E][Mc^2 + E - \Delta(r)]\right) G_{n_r.k}(r) = 0 \quad \ldots (28)$$

Under the condition of the pseudospin symmetry sum of the potentials $V(r)$ and $S(r)$ can be written as:

$$\Delta(r) = 8q_1 \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} - 2q_2 \frac{(1 + e^{-2\alpha r})}{(1 - e^{-2\alpha r})} + \frac{2v_0}{(1 - e^{-2\alpha r})} - \frac{2v_1}{(1 - e^{-2\alpha r})^2} \quad \ldots (29)$$

By substituting Eq. (29) into Eq. (28), we obtain the upper radial equation of Dirac equation as:

$$\left\{\frac{d^2}{dr^2} + \frac{(E^2 - M^2c^4)}{\hbar^2 c^2} - \frac{(E - Mc^2)}{\hbar^2 c^2}\left[8q_1 \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2} - 2q_2 \frac{(1 + e^{-2\alpha r})}{(1 - e^{-2\alpha r})} + \frac{2v_0}{(1 - e^{-2\alpha r})} - \frac{2v_1}{(1 - e^{-2\alpha r})^2}\right] - \frac{k(k-1)}{r^2}\right\} G_{n_r.k}(r) = 0 \quad \ldots (30)$$

By using the pekeris-type approximation and using the transformation $s = \exp(-2\alpha r)$ we can write the Eq. (30) as summarized below:

$$G''_{n_r.k}(s) + \frac{(1-s)}{s(1-s)} G'_{n_r.k}(s) + \frac{1}{s^2(1-s)^2}[\eta'_2 s^2 + \eta'_1 s + \eta'_0] G_{n_r.k}(s) = 0 \quad \ldots (31)$$

Where the parameters η'2, η'1 and η'0 are considered as follows:

$$\begin{cases} \eta_2' = \frac{\delta}{4\alpha^2} - \frac{\gamma'}{2\alpha^2} q_2 \\ \eta_1' = -\frac{\delta}{2\alpha^2} + \frac{\gamma'}{2\alpha^2}(v_0 - 4q_1) - k(k-1) \\ \eta_0' = \frac{\delta}{4\alpha^2} + \frac{\gamma'}{2\alpha^2}(q_2 - v_0 + v_1) \end{cases} \quad \text{... (32)}$$

where $\delta = \frac{(E^2 - M^2c^4)}{\hbar^2 c^2}$ and $\gamma' = \frac{(E - Mc^2)}{\hbar^2 c^2}$

Now by comparing Eq. (31) with Eq. (1), we can easily obtain the coefficients k_i (i = 1, 2, 3) as follows:

$$k_1 = k_2 = k_3 = 1 \quad \text{... (33)}$$

The values of the coefficients k_i (i = 4, 5) are also found from Eq. (5) as below:

$$\begin{cases} k_4 = \sqrt{-\eta_0'} \\ k_5 = \frac{1}{2} + \sqrt{\frac{1}{4} - [\eta_2' + \eta_1' + \eta_0']} \end{cases} \quad \text{... (34)}$$

Thus, by the use of energy equation Eq. (2) for energy Eigen-values, we find:

$$(2n+1) + \sqrt{\frac{2\gamma'}{\alpha^2}(v_0 - q_2 - v_1) - \frac{\delta}{\alpha^2}} + \sqrt{(2k-1)^2 - \frac{2\gamma'}{\alpha^2}(v_1 - 4q_1)} - \sqrt{\frac{2\gamma'}{\alpha^2} q_2 - \frac{\delta}{\alpha^2}} = 0 \quad \text{... (35)}$$

And using Eq. (32) we can obtain the energy Eigen-values equation, in closed form, as:

$$(2n+1) + \frac{\sqrt{(E - Mc^2)}}{\hbar c \alpha}\left[\sqrt{2(v_0 - v_1 - q_2) - (E + Mc^2)} - \sqrt{2q_2 - (E + Mc^2)}\right] + \sqrt{(2k-1)^2 - \frac{2(E - Mc^2)}{\hbar^2 c^2 \alpha^2}(v_1 - 4q_1)} = 0 \quad \text{... (36)}$$

Using Eq. (13) and Eq. (34) we can finally obtain the wave functions as below:

$$G_{n_r.k}(r) = N'(e^{-2\alpha r})^{\left(\sqrt{-\eta_0'}\right)}(1 - e^{-2\alpha r})^{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \eta_2' + \eta_1' + \eta_0'}\right)} {}_2F_1\left(-n. n + 2\left(\sqrt{-\eta_0'} + \frac{1}{2} + \sqrt{\frac{1}{4} + \eta_2' + \eta_1' + \eta_0'}\right); 2\sqrt{-\eta_0'} + 1. e^{-2\alpha r}\right) \quad \text{... (37)}$$

Where N' is the normalization constant, on the other hand, the upper component of the Dirac spinor can be calculated by Eq. (38) as:

$$F_{n_r.k}(r) = \frac{\hbar^2 c^2}{(Mc^2 - E)}\left(\frac{d}{dr} - \frac{k}{r}\right) G_{n_r.k}(r) \quad \text{... (38)}$$

Different between spin symmetry and pseudospin symmetry can be written as:

$$\begin{cases} (E + Mc^2) \Leftrightarrow (E - Mc^2) \\ k(k+1) \Leftrightarrow k(k-1) \end{cases} \quad \text{... (39)}$$

Finally, we plot the relativistic energy Eigen-values of the Eckart plus Hulthen potentials with spin and pseudospin symmetry limitations in Figs 3 and 4. In these figures, we plot the energy Eigen-values of spin and p-spin symmetry limits versus potential parameters α, v_0 and q_2.

In Figs 2 and 3 under the condition of the (a) spin and (b) pseudospin symmetries we show the behavior

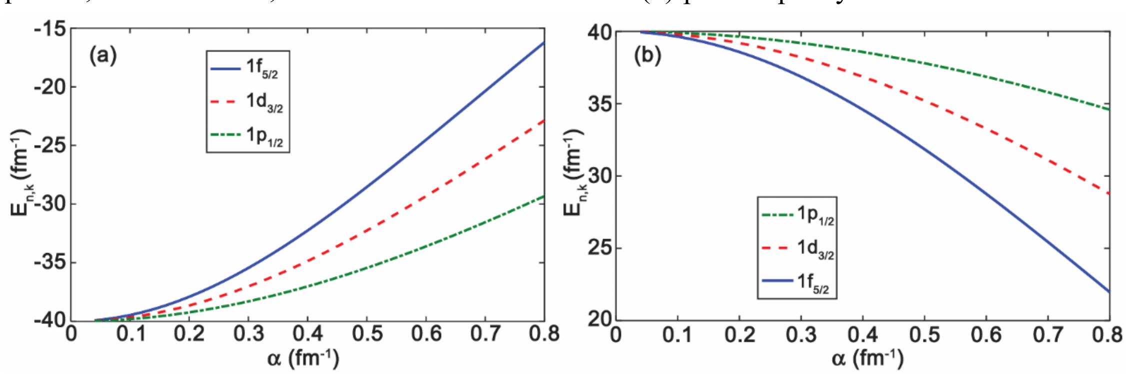

Fig. 2 — Energy spectra in the (a) spin and (b) pseudospin symmetries at various α (fm^{-1}) with parameters $\hbar$=c=1, v_0=2 fm^{-1}, v_1=4 fm^{-1}, q_1=1 fm^{-1}, q_2=5 fm^{-1}, m=40 fm^{-1}.

of the energy Eigen-values equation for three levels energy for some of screening parameter values (α and v_0) by taking the strength parameters $\hbar=c=1$, $v_0=2$ fm^{-1}, $v_1=4$ fm^{-1}, $q_1=1$ fm^{-1}, $q_2=5$ fm^{-1}, $m=40$ fm^{-1}. It is seen that the energy Eigen-values increase for spin symmetry and decrease for pseudospin symmetry with the increasing of the screening parameter value. But in Fig. 4, under the condition of the (a) spin and (b) pseudospin symmetries, it is seen that the energy Eigen-values decrease for spin symmetry and increase for pseudospin symmetry with the increasing of the screening parameter value (q_2).

5 Some Special Cases

In this section we consider some special cases of interest if we consider $q_1=v_1=0$ the modified Eckart plus Hulthen potentials can be written:

$$V(r) = -q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})} + \frac{v_0}{(1-e^{-2\alpha r})} \quad \text{... (40)}$$

For spin symmetry the Dirac equation can be written as:

$$\left(-\frac{d^2}{dr^2} + \frac{k(k+1)}{r^2} + \frac{1}{\hbar^2c^2}[Mc^2+E][Mc^2-E+\Sigma(r)]\right)F_{n_r,k}(r) = 0 \quad \text{... (41)}$$

And,

$$\Sigma(r) = -2q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})} + \frac{2v_0}{(1-e^{-2\alpha r})} \quad \text{... (42)}$$

By substituting Eq. (42) into Eq. (41), we obtain the upper radial equation of Dirac equation as:

$$\left\{\frac{d^2}{dr^2} + \frac{(E^2-M^2c^4)}{\hbar^2c^2} - \frac{(E+Mc^2)}{\hbar^2c^2}\left[-2q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})} + \frac{2v_0}{(1-e^{-2\alpha r})}\right] - \frac{k(k+1)}{r^2}\right\}F_{n_r,k}(r) = 0 \quad \text{... (43)}$$

Equation (43) is exactly solvable only for the case of k = 0,-1. In order to obtain the analytical solutions of Eq. (43), we employ the improved pekeris approximation and replace the spin–orbit coupling term with the expression that is valid[46] for $\alpha r << 1$.

$$\frac{k(k+1)}{r^2} \approx \frac{k(k+1)4\alpha^2e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \quad \text{... (44)}$$

Using the transformation $s=\exp(-2\alpha r)$ Eq. (43) brings into the form:

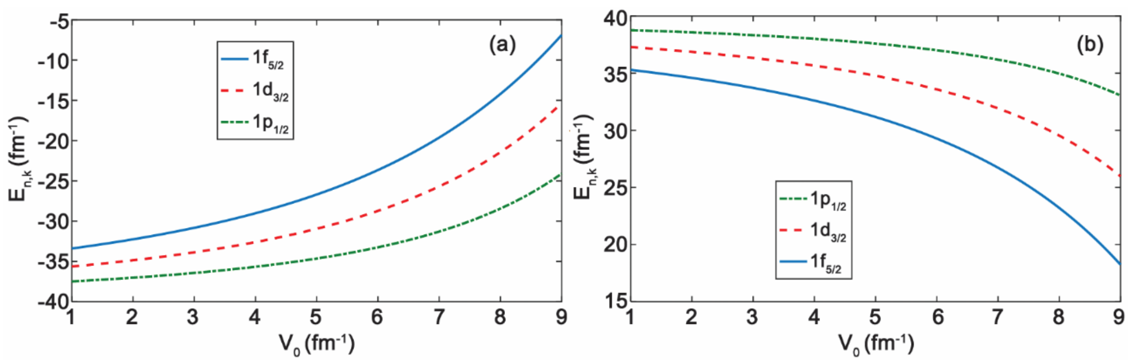

Fig. 3 — Energy spectra in the (a) spin and (b) pseudospin symmetries at various v_0 (fm^{-1}) with parameters $\hbar=c=1$, α=0.4fm^{-1}, v_1=4fm^{-1}, q_1=1 fm^{-1}, q_2=5 fm^{-1}, m=40 fm^{-1}.

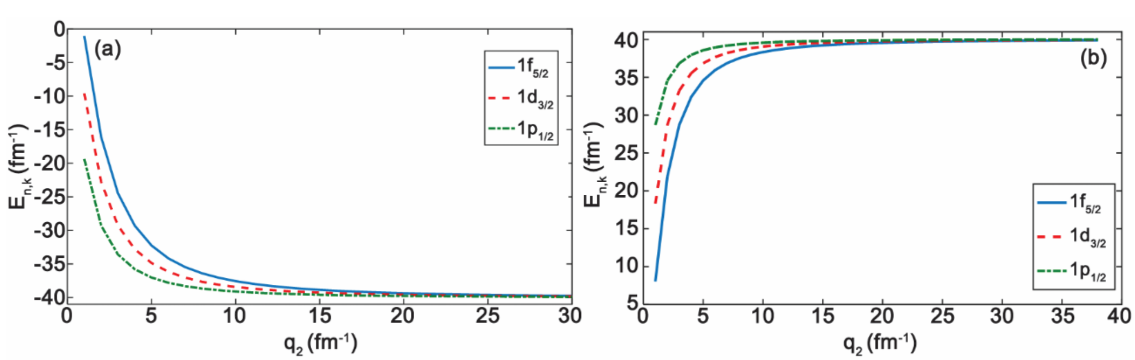

Fig. 4 – Energy spectra in the (a) spin and (b) pseudospin symmetries at various q_2 (fm^{-1}) with parameters $\hbar=c=1$, α=0.4 fm^{-1}, v_1=4 fm^{-1}, q_1=1 fm^{-1}, v_0=2 fm^{-1}, m=40 fm^{-1}.

$$F''_{n_r.k}(s)+\frac{(1-s)}{s(1-s)}F'_{n_r.k}(s)+\frac{1}{s^2(1-s)^2}[\eta_2 s^2+\eta_1 s+\eta_0]F_{n_r.k}(s)=0 \qquad \text{... (45)}$$

Where the parameters η_2, η_1 and η_0 are considered as follows:

$$\begin{cases} \eta_2=\frac{\delta}{4\alpha^2}-\frac{\gamma}{2\alpha^2}q_2 \\ \eta_1=-\frac{\delta}{2\alpha^2}+\frac{\gamma}{2\alpha^2}(v_0)\ -k(k+1) \\ \eta_0=\frac{\delta}{4\alpha^2}+\frac{\gamma}{2\alpha^2}(q_2-v_0) \end{cases} \qquad \text{... (46)}$$

Where, $\delta=\frac{(E^2-M^2c^4)}{\hbar^2c^2}$ and $\gamma=\frac{(E+Mc^2)}{\hbar^2c^2}$

Now by comparing Eq. (45) with Eq. (1), we can easily obtain the coefficients k_i (i = 1, 2, 3) as follows:

$$k_1=k_2=k_3=1 \qquad \text{... (47)}$$

The values of the coefficients k_i (i = 4, 5) are also found from Eq. (5) as below:

$$\begin{cases} k_4=\sqrt{-\eta_0} \\ k_5=\frac{1}{2}+\sqrt{\frac{1}{4}-[\eta_2+\eta_1+\eta_0]} \end{cases} \qquad \text{... (48)}$$

Thus, by the use of energy equation (Eq. (2)) for energy Eigen-values, we find:

$$(2n+1)+\sqrt{\frac{2\gamma}{\alpha^2}(v_0-q_2)-\frac{\delta}{\alpha^2}}-\sqrt{\frac{2\gamma}{\alpha^2}q_2-\frac{\delta}{\alpha^2}}+\sqrt{(2k+1)^2}=0 \qquad \text{... (49)}$$

We can obtain the energy Eigen-values equation, in closed form, as:

$$(2n+1)+\frac{\sqrt{(E+Mc^2)}}{\hbar c\alpha}\left[\sqrt{2(v_0-q_2)-(E-Mc^2)}-\sqrt{2q_2-(E-Mc^2)}\right]+\sqrt{(2k+1)^2}=0 \qquad \text{... (50)}$$

By using Eq. (4), we can finally obtain the wave functions with the aid of Eq. (48) as:

$$F_{n_r.k}(r)=N(e^{-2\alpha r})^{(\sqrt{-\eta_0})}(1-e^{-2\alpha r})^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\eta_2+\eta_1+\eta_0}\right)}\ {}_2F_1\left(-n.n+2\left(\sqrt{-\eta_0}+\frac{1}{2}+\sqrt{\frac{1}{4}+\eta_2+\eta_1+\eta_0}\right);2\sqrt{-\eta_0}+1.e^{-2\alpha r}\right) \qquad \text{... (51)}$$

Where N is the normalization constant and, the lower component of the Dirac spinor can be calculated by Eq. (52) as:

$$G_{n_r.k}(r)=\frac{\hbar^2c^2}{(E+Mc^2)}\left(\frac{d}{dr}+\frac{k}{r}\right)F_{n_r.k}(r) \qquad \text{... (52)}$$

We know that different between spin symmetry and pseudospin symmetry can be written as:

$$\begin{cases} (E+Mc^2)\Leftrightarrow(E-Mc^2) \\ k(k+1)\Leftrightarrow k(k-1) \end{cases} \qquad \text{... (53)}$$

By using Eq. (53) we could have obtained energy Eigen-values and the wave function of the radial Dirac equation for the pseudospin symmetry as:

$$(2n+1)+\sqrt{\frac{2\gamma'}{\alpha^2}(v_0-q_2)-\frac{\delta}{\alpha^2}}-\sqrt{\frac{2\gamma'}{\alpha^2}q_2-\frac{\delta}{\alpha^2}}+\sqrt{(2k-1)^2}=0 \qquad \text{... (54)}$$

We can obtain the energy Eigen-values equation, in closed form as:

$$(2n+1)+\frac{\sqrt{(E-Mc^2)}}{\hbar c\alpha}\left[\sqrt{2(v_0-q_2)-(E+Mc^2)}-\sqrt{2q_2-(E+Mc^2)}\right]+\sqrt{(2k-1)^2}=0 \qquad \text{... (55)}$$

Also we can finally obtain the wave functions as below:

$$G_{n_r.k}(r)=N'(e^{-2\alpha r})^{\left(\sqrt{-\eta'_0}\right)}(1-e^{-2\alpha r})^{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+\eta'_2+\eta'_1+\eta'_0}\right)}\ {}_2F_1\left(-n.n+2\left(\sqrt{-\eta'_0}+\frac{1}{2}+\sqrt{\frac{1}{4}+\eta'_2+\eta'_1+\eta'_0}\right);2\sqrt{-\eta'_0}+1.e^{-2\alpha r}\right) \qquad \text{... (56)}$$

Where N' is the normalization constant, on the other hand, the upper component of the Dirac spinor can be calculated by Eq. (57) as:

$$F_{n_r.k}(r)=\frac{\hbar^2c^2}{(Mc^2-E)}\left(\frac{d}{dr}-\frac{k}{r}\right)G_{n_r.k}(r) \qquad \text{... (57)}$$

We have obtained the energy Eigen-values and the wave function of the radial Dirac equation for spatially modified Eckart plus Hulthen potentials with the pseudospin symmetry for $k\neq 0$.

6 Conclusions

In this paper, we have discussed analytically the solutions of the Dirac equation for Eckart plus Hulthen potentials with Spin Symmetry and Pseudospin Symmetry for $k \neq 0$. We could obtain the energy Eigen-values and the wave function in terms of the generalized Laguerre polynomial functions via the formula method. We have also considered the limiting cases of spin and pseudo spin symmetry for modified Eckart plus Hulthen potentials to obtain the energy Eigen-values and the wave function. To show the accuracy of the present model, some numerical values of the energy levels are shown in Figs 2-4. We can conclude that our results are interesting for experimental physicists, because the results more general and useful to study nuclear scattering, nuclear and particle physics.

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